



A New Compact Numerical Scheme for Solving Time Fractional Mobile-Immobile Advection-Dispersion Model

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Abstract

This work is focused on the derivation and analysis of a novel numerical technique for solving time fractional mobile-immobile advection-dispersion equation which models many complex systems in engineering and science. The scheme is derived using the effective combination of Euler and Caputo numerical techniques for approximating the integer and fractional time derivatives respectively, and a fourth order exponential compact scheme for spatial derivatives. The Fourier analysis technique is used to prove that the proposed numerical scheme is unconditionally stable and perform convergence analysis. To assess the viability and accuracy of the proposed scheme, some numerical examples are demonstrated with constant as well as variable order time fractional derivatives for this model.

Keywords: time fractional; advection-dispersion; mobile-immobile; stability; convergence; computational order.

1 Introduction

The fractional calculus is one of the most significant and useful generalisations of the conventional derivatives of integer orders. For many complicated systems that fluctuate in space and time, the fractional-order partial differential equations can offer more accurate and realistic description than classical integer-order partial differential equations. Fractional calculus has recently proven to be useful in mimicking aberrant behaviours that can occur in a range of scientific and engineering fields. Many mathematical models are developed using the fractional derivatives like the nonlinear behaviour of earthquake [6], continuum fluid-dynamics [7], porous media [8], hydrology [18], fractured media at regional scales [40], complex dynamics in biological tissues [17], non-Fickian transport [37, 38], optimization of radiotherapy cancer treatments [5], SIQ mathematical model of Corona virus disease [23].

Moreover, financial problems like Black-Scholes option pricing were modelled using fractional order differential equations in [10, 24]. The development of effective and trustworthy numerical approaches for fractional differential equations (FDEs) is of tremendous interest because there are just a handful of basic examples for which analytic solutions to FDEs are available. There have been many numerical methods put out so far to solve FDEs. In that context, finite difference techniques for time-fractional convection diffusion equations (TFCDEs) play a significant role. For example, finite difference methods for time fractional diffusion problems were studied in [11, 13] which are quadratic convergent in space. A review of the finite difference methods for solving the FDEs have been done in [12].

Murio used Caputo's fractional approximation in [21] and developed an implicit numerical scheme for solving time fractional linear diffusion problem which is unconditionally stable. Mohebbi and Abbaszadeh [20] proposed a fourth order in space unconditionally stable method for time fractional advection-dispersion equation. Cui [3] proposed an implicit scheme for TFCDEs, later the same author extended the scheme to the TFCDE with reaction term and with variable coefficients [4]. The authors of [34] have proposed a discrete form for solving TFCDE. Recently, numerical methods for fractional Black-Scholes option pricing, which is modelled as a TFCDE, were developed in [1, 25]. Integer order steady and unsteady convection-diffusion equations have been the subject of numerous research works since many years. Among that [28, 29] highlights the use of the high-order compact exponential technique for these types of equations. Therefore, it is wise to use these compact exponential schemes to solve convection-diffusion problems with fractional order time derivative [3, 4].

A fractional mobile-immobile transport model is a type of second-order time-fractional partial differential equation representing a variety of problems in time-varying physical and mathematical systems. In order to transport solutes in the overall network, the connectedness and heterogeneity of spatial properties can be better described by the transportation model. Mobile-immobile models have been widely praised by hydrologists studying water transport in saturated and unsaturated regions [2]. Schumer et al. [26] modelled the fractional mobile-immobile convection-diffusion process for the total solute transport. Zhang et al. [36] applied this model to simulate solute concentrations for an experiment near the River Dee. In recent decades, several works have been devoted to the derivation of the numerical schemes and applications of the fractional order mobile-immobile equation [30, 39].

To the author's best knowledge, there aren't many works that discuss the numerical schemes and theoretical analysis of the aforementioned solute transport fractional mobile-immobile mathematical model. For the temporal fractional mobile-immobile transport model, Liu et al. [15] provide a finite difference approach and a meshless method. The stability and convergence of the

scheme were also covered in the paper. A numerical simulation of the fractional mobile-immobile advection-dispersion model is taken into account by Pourbashash [22]. The proposed method in [22] is based on Legendre spectral method in space and finite differences for time. Two numerical schemes are derived in [32] to solve this model. An implicit Euler approximation and its convergence in a special case has been described in [33]. A finite difference Crank–Nicolson method with stability and convergence results are given in Liu and Li [16] for solving this time variable fractional order model equation. ADI schemes for time fractional mobile–immobile linear and semi linear diffusion equations were proposed in [9, 31]. Most recently, numerical schemes and analysis based on finite element method (FEM) were established in [19, 35] for solving variable order time fractional option pricing models. A characteristic FEM for solving TFCDE was developed in [14].

In this work, we construct a new higher order numerical technique based on a uniform discretization to solve variable order time fractional mobile-immobile advection–dispersion model subject to initial and boundary conditions. This scheme uses the fourth order compact exponential method to approximate spatial derivatives, while backward Euler derivative and Caputo fractional derivative respectively are used to approximate integer and fractional temporal derivatives. The proposed compact scheme gives high accurate results and is relatively easy to implement compared to FEM in solving the model under consideration. Using Fourier analysis technique, it is proven that our method is unconditionally stable. The uniqueness and convergence analysis of the new constructed method is performed. Some test problems are considered to demonstrate the convergence and accuracy of the new method.

This paper is structured as follows. In Section 2, we introduce a compact exponential difference scheme for solving time-fractional order mobile-immobile transport equation by first applying a fourth order exponential compact scheme for the steady state problem, and then applying the backward Euler and Caputo’s derivative for the integer and fractional derivative of temporal variable respectively. In Section 3, the uniqueness of the solution is discussed. Also the convergence and stability are proved using the Fourier analysis. In the next Section, some computational results using the new scheme are provided to support the theoretical findings. Conclusions are deduced in the end.

2 Construction of the Numerical Scheme

In this section, we derive a new numerical method to solve the time fractional mobile-immobile convection-diffusion equation. Let $\Omega = 0 < x < L$, this equation is given by

$$\frac{\partial y}{\partial t} + b \frac{\partial^\gamma y}{\partial t^\gamma} + c \frac{\partial y}{\partial x} = d \frac{\partial^2 y}{\partial x^2} + f(x, t), \quad \text{in } \Omega \times (0, T]. \quad (1)$$

The initial condition is

$$y(x, 0) = \psi(x), \quad \text{in } \Omega,$$

and the Dirichlet boundary conditions are

$$y(0, t) = \phi_1(t), \quad y(L, t) = \phi_2(t), \quad 0 \leq t \leq T,$$

where y represents a scalar variable in x and t with constant velocity c ($\neq 0$) and constant diffusivity $d > 0$. To derive the scheme for (1), first we consider the steady state equation

$$c \frac{\partial y}{\partial x} - d \frac{\partial^2 y}{\partial x^2} = F \quad \text{in } \Omega. \quad (2)$$

Let the space interval $[0, L]$ be partitioned uniformly with step size $h = x_{i+1} - x_i$, $i = 0, 1, \dots, M-1$ and the time interval $[0, T]$ is divided uniformly with step size τ , thus $t_n = n\tau$ is the time level of n^{th} step. A fourth order exponential compact scheme for (2), see [29] can be obtained as

$$-\beta\delta_x^2 y_i + c\delta_x y_i = F_i + \beta_1\delta_x F_i + \beta_2\delta_x^2 F_i, \quad (3)$$

where

$$\beta = \frac{ch}{2} \coth\left(\frac{ch}{2d}\right), \quad \beta_1 = \frac{d-\beta}{c}, \quad \beta_2 = \frac{d(d-\beta)}{c^2} + \frac{h^2}{6},$$

and δ_x and δ_x^2 , for $x \in (x_{i-1}, x_{i+1})$, $i = 1, 2, \dots, M-1$ are the standard second order central difference operators for first and second derivatives respectively. Substituting $F = -\frac{\partial y}{\partial t} - b\frac{\partial^\gamma y}{\partial t^\gamma} + f$ in equation (3) and arranging the terms, a fourth order compact exponential semi-discrete approximation for the unsteady 1D mobile-immobile convection-diffusion problem (1) is developed as follows:

$$\begin{aligned} & \left(\frac{\beta_2}{h^2} - \frac{\beta_1}{2h}\right) \left(\frac{\partial y}{\partial t} + b\frac{\partial^\gamma y}{\partial t^\gamma} - f\right)_{i-1}^n + \left(1 - \frac{2\beta_2}{h^2}\right) \left(\frac{\partial y}{\partial t} + b\frac{\partial^\gamma y}{\partial t^\gamma} - f\right)_i^n \\ & + \left(\frac{\beta_2}{h^2} + \frac{\beta_1}{2h}\right) \left(\frac{\partial y}{\partial t} + b\frac{\partial^\gamma y}{\partial t^\gamma} - f\right)_{i+1}^n = \left(\frac{\beta}{h^2} + \frac{c}{2h}\right) y_{i-1}^n - \frac{2\beta}{h^2} y_i^n + \left(\frac{\beta}{h^2} - \frac{c}{2h}\right) y_{i+1}^n. \end{aligned} \quad (4)$$

The Caputo derivative to approximate the fractional order time derivative of $O(\tau^{2-\gamma})$ is given in [27]. The proof of the below Lemma is given in [27].

Lemma 2.1. Let $g(t) \in C^2[0, t_n]$ and

$$\begin{aligned} \bar{R}(g(t_n)) &:= \frac{1}{\Gamma(1-\gamma)} \int_0^{t_n} \frac{g'(s)}{(t_n-s)^\gamma} ds \\ &- \frac{\tau^{-\gamma}}{\Gamma(2-\gamma)} \left[a_0 g(t_n) - \sum_{j=1}^{n-1} (a_{n-j-1} - a_{n-j}) g(t_j) - a_{n-1} g(t_0) \right], \end{aligned}$$

then,

$$|\bar{R}(g(t_n))| \leq \frac{1}{2\Gamma(1-\gamma)} \left[\frac{1}{4} + \frac{\gamma}{(1-\gamma)(2-\gamma)} \right] \max_{0 \leq t \leq t_n} |g''(t)| \tau^{2-\gamma},$$

for $\gamma \in (0, 1)$ and $a_j = (j+1)^{1-\gamma} - j^{1-\gamma}$. Then, the fractional derivative can be approximated by

$${}_0^C D_t^\gamma y(x_j, t_n) = \frac{\tau^{-\gamma}}{\Gamma(2-\gamma)} \left[a_0 y_j^n - \sum_{l=1}^{n-1} (a_{n-l-1} - a_{n-l}) y_j^l - a_{n-1} y_j^0 \right] + O(\tau^{2-\gamma}). \quad (5)$$

Lemma 2.2. The coefficients $a_j = (1+j)^{1-\gamma} - j^{1-\gamma}$, $j = 0, 1, 2, \dots$, satisfy

$$(i) \quad 1 = a_0 > a_1 > a_2 > \dots > a_j > \dots \rightarrow 0,$$

$$(ii) \quad (1-\gamma)(1+n)^{-\gamma} < a_n < (1-\gamma)n^{-\gamma}.$$

The time derivatives of integer and fractional order in (4) are replaced with backward Euler and

Caputo derivative (5) respectively. After rearranging the terms, the scheme for (1) is obtained as

$$\begin{aligned} & \left[\left(\frac{1}{\tau} + \mu \right) \left(\frac{\beta_2}{h^2} - \frac{\beta_1}{2h} \right) - \left(\frac{\beta}{h^2} + \frac{c}{2h} \right) \right] y_{i-1}^1 + \left[\left(\frac{1}{\tau} + \mu \right) \left(1 - \frac{2\beta_2}{h^2} \right) + \frac{2\beta}{h^2} \right] y_i^1 \\ & + \left[\left(\frac{1}{\tau} + \mu \right) \left(\frac{\beta_2}{h^2} + \frac{\beta_1}{2h} \right) - \left(\frac{\beta}{h^2} - \frac{c}{2h} \right) \right] y_{i+1}^1 = \frac{1}{\tau} \left[\left(\frac{\beta_2}{h^2} - \frac{\beta_1}{2h} \right) y_{i-1}^0 + \left(1 - \frac{2\beta_2}{h^2} \right) y_i^0 \right. \\ & + \left. \left(\frac{\beta_2}{h^2} + \frac{\beta_1}{2h} \right) y_{i+1}^0 \right] + \mu \left[\left(\frac{\beta_2}{h^2} - \frac{\beta_1}{2h} \right) y_{i-1}^0 + \left(1 - \frac{2\beta_2}{h^2} \right) y_i^0 + \left(\frac{\beta_2}{h^2} + \frac{\beta_1}{2h} \right) y_{i+1}^0 \right] \\ & + \left[\left(\frac{\beta_2}{h^2} - \frac{\beta_1}{2h} \right) f_{i-1}^1 + \left(1 - \frac{2\beta_2}{h^2} \right) f_i^1 + \left(\frac{\beta_2}{h^2} + \frac{\beta_1}{2h} \right) f_{i+1}^1 \right], \quad 1 \leq i \leq M-1, \end{aligned} \quad (6)$$

and

$$\begin{aligned} & \left[\left(\frac{1}{\tau} + \mu \right) \left(\frac{\beta_2}{h^2} - \frac{\beta_1}{2h} \right) - \left(\frac{\beta}{h^2} + \frac{c}{2h} \right) \right] y_{i-1}^n + \left[\left(\frac{1}{\tau} + \mu \right) \left(1 - \frac{2\beta_2}{h^2} \right) + \frac{2\beta}{h^2} \right] y_i^n \\ & + \left[\left(\frac{1}{\tau} + \mu \right) \left(\frac{\beta_2}{h^2} + \frac{\beta_1}{2h} \right) - \left(\frac{\beta}{h^2} - \frac{c}{2h} \right) \right] y_{i+1}^n = \frac{1}{\tau} \left[\left(\frac{\beta_2}{h^2} - \frac{\beta_1}{2h} \right) y_{i-1}^{n-1} + \left(1 - \frac{2\beta_2}{h^2} \right) y_i^{n-1} \right. \\ & + \left. \left(\frac{\beta_2}{h^2} + \frac{\beta_1}{2h} \right) y_{i+1}^{n-1} \right] + \mu \sum_{l=1}^{n-1} (a_{n-l-1} - a_{n-l}) \left[\left(\frac{\beta_2}{h^2} - \frac{\beta_1}{2h} \right) y_{i-1}^l + \left(1 - \frac{2\beta_2}{h^2} \right) y_i^l \right. \\ & + \left. \left(\frac{\beta_2}{h^2} + \frac{\beta_1}{2h} \right) y_{i+1}^l \right] + \mu a_{n-1} \left[\left(\frac{\beta_2}{h^2} - \frac{\beta_1}{2h} \right) y_{i-1}^0 + \left(1 - \frac{2\beta_2}{h^2} \right) y_i^0 + \left(\frac{\beta_2}{h^2} + \frac{\beta_1}{2h} \right) y_{i+1}^0 \right] \\ & + \left[\left(\frac{\beta_2}{h^2} - \frac{\beta_1}{2h} \right) f_{i-1}^n + \left(1 - \frac{2\beta_2}{h^2} \right) f_i^n + \left(\frac{\beta_2}{h^2} + \frac{\beta_1}{2h} \right) f_{i+1}^n \right], \quad 1 \leq i \leq M-1, \quad 2 \leq n \leq N, \end{aligned} \quad (7)$$

where $\mu = b \frac{\tau^{-\gamma}}{\Gamma(2-\gamma)}$. The solution to the linear system (6) and (7) for $0 < \gamma(x, t) < 1$, gives the solution to equation (1).

3 Mathematical Analysis of the Scheme

In this section we provide rigorous theoretical analysis of the numerical scheme derived in the previous section. First, we prove the uniqueness of the solution followed by stability analysis and convergence.

3.1 Uniqueness of the solution

Theorem 3.1. *The higher order exponential scheme (6)-(7) has a unique solution.*

Proof. The scheme (6)-(7) can be expressed in the following matrix form as

$$AY^n = \sum_{l=0}^{n-1} B_l Y^l + C^n,$$

where $A = \text{tridiag}(L, D, U)$, with $L = \left[\left(\frac{1}{\tau} + \mu \right) \left(\frac{\beta_2}{h^2} - \frac{\beta_1}{2h} \right) - \left(\frac{\beta}{h^2} + \frac{c}{2h} \right) \right]$,
 $D = \left[\left(\frac{1}{\tau} + \mu \right) \left(1 - \frac{2\beta_2}{h^2} \right) + \frac{2\beta}{h^2} \right]$ and $U = \left[\left(\frac{1}{\tau} + \mu \right) \left(\frac{\beta_2}{h^2} + \frac{\beta_1}{2h} \right) - \left(\frac{\beta}{h^2} - \frac{c}{2h} \right) \right]$.

Note that,

$$\frac{\beta}{h^2} \pm \frac{c}{2h} = \frac{d}{h^2} \left(\frac{ch}{2d} \coth \left(\frac{ch}{2d} \right) \pm \frac{ch}{2d} \right) = \frac{d}{h^2} \frac{ch}{2d} \left(\coth \left(\frac{ch}{2d} \right) \pm 1 \right).$$

If we let $u = \frac{ch}{2d}$, then,

$$\frac{ch}{2d} \left(\coth \left(\frac{ch}{2d} \right) \pm 1 \right) = 2u \frac{e^{\pm u}}{e^u - e^{-u}} = 2e^{u \pm u} \frac{u}{e^{2u} - 1} \geq 0, \quad u \in (-\infty, +\infty).$$

Hence,

$$\left| \frac{\beta}{h^2} + \frac{c}{2h} \right| + \left| \frac{\beta}{h^2} - \frac{c}{2h} \right| = \left| \frac{2\beta}{h^2} \right| = \frac{2\beta}{h^2}.$$

Using Lemma 3 in [29], we get

$$1 - \frac{2\beta_2}{h^2} = \left| 1 - \frac{2\beta_2}{h^2} \right| > \left| \frac{\beta_2}{h^2} + \frac{\beta_1}{2h} \right| + \left| \frac{\beta_2}{h^2} - \frac{\beta_1}{2h} \right|.$$

Therefore,

$$\begin{aligned} |L| + |U| &\leq \left| \frac{\beta_2}{h^2} + \frac{\beta_1}{2h} \right| + \left| \frac{\beta_2}{h^2} - \frac{\beta_1}{2h} \right| + \left(\frac{1}{\tau} + \mu \right) \left(\left| \frac{\beta}{h^2} + \frac{c}{2h} \right| + \left| \frac{\beta}{h^2} - \frac{c}{2h} \right| \right) \\ &< \left(\frac{1}{\tau} + \mu \right) \left(1 - \frac{2\beta_2}{h^2} \right) + \left| \frac{2\beta}{h^2} \right| = \left(\frac{1}{\tau} + \mu \right) \left(1 - \frac{2\beta_2}{h^2} \right) + \frac{2\beta}{h^2} = |D|. \end{aligned} \quad (8)$$

This implies the matrix A is strictly diagonally dominant, therefore A is nonsingular. Hence, there exist a unique solution. $\square$

3.2 Stability analysis

In this section, we rigorously analyze the stability of the proposed scheme (6)-(7) by the aid of Fourier analysis method. Firstly, we denote

$$\begin{aligned} v_1 &= \left(\frac{1}{\tau} + \mu \right) \left(\frac{\beta_2}{h^2} + \frac{\beta_1}{2h} \right) - \left(\frac{\beta}{h^2} - \frac{c}{2h} \right), & v_2 &= \left(\frac{1}{\tau} + \mu \right) \left(1 - \frac{2\beta_2}{h^2} \right) + \frac{2\beta}{h^2}, \\ v_3 &= \left(\frac{1}{\tau} + \mu \right) \left(\frac{\beta_2}{h^2} - \frac{\beta_1}{2h} \right) - \left(\frac{\beta}{h^2} + \frac{c}{2h} \right), & w_1 &= \frac{\beta_2}{h^2} + \frac{\beta_1}{2h}, \\ w_2 &= 1 - \frac{2\beta_2}{h^2}, & \text{and} & \quad w_3 = \frac{\beta_2}{h^2} - \frac{\beta_1}{2h}. \end{aligned} \quad (9)$$

Let $\tilde{Y}_j^n$ be the approximation to the solution of (6)-(7) and define

$$\rho_j^n = Y_j^n - \tilde{Y}_j^n, \quad 1 \leq j \leq M-1, \quad 0 \leq n \leq N,$$

with corresponding vector $\rho^n = (\rho_1^n, \rho_2^n, \dots, \rho_{M-1}^n)^T$. Then, for $n = 1$, we have

$$v_1 \rho_{j+1}^1 + v_2 \rho_j^1 + v_3 \rho_{j-1}^1 = \left(\frac{1}{\tau} + \mu \right) (w_1 \rho_{j+1}^0 + w_2 \rho_j^0 + w_3 \rho_{j-1}^0), \quad (10)$$

and for $n \geq 2$, we have

$$\begin{aligned} v_1 \rho_{j+1}^n + v_2 \rho_j^n + v_3 \rho_{j-1}^n &= \frac{1}{\tau} (w_1 \rho_{j+1}^{n-1} + w_2 \rho_j^{n-1} + w_3 \rho_{j-1}^{n-1}) \\ &\quad + \mu \sum_{l=1}^{n-1} (a_{n-l-1} - a_{n-l}) (w_1 \rho_{j+1}^l + w_2 \rho_j^l + w_3 \rho_{j-1}^l) \\ &\quad + \mu a_{n-1} (w_1 \rho_{j+1}^0 + w_2 \rho_j^0 + w_3 \rho_{j-1}^0). \end{aligned} \quad (11)$$

We extend ρ_j^n from the node x_j to the interval $(x_{j-\frac{h}{2}}, x_{j+\frac{h}{2}})$ using piecewise constants, thus we define $\rho^n(x)$ in the whole domain $(0, L)$. Here $\rho^n(x) = 0$ on the boundaries because of the Dirichlet boundary conditions, therefore we define

$$\rho^n(x) = \begin{cases} \rho_j^n, & \text{in } x_{j-\frac{h}{2}} < x \leq x_{j+\frac{h}{2}}, \quad 1 \leq j \leq M-1, \\ 0, & \text{in } 0 \leq x \leq \frac{h}{2} \text{ and } L - \frac{h}{2} < x \leq L. \end{cases}$$

The Fourier series for $\rho^n(x)$ can be expressed as $\rho^n(x) = \sum_{l=-\infty}^{\infty} \xi^n(l) e^{2\pi i l x / L}$. For any vector

$y = (y_1, y_2, \dots, y_{M-1}) \in \mathbb{R}^{M-1}$, the discrete l^2 norm is considered as $\|y\|_{l^2} = \left(h \sum_{j=1}^{M-1} y_j^2 \right)^{\frac{1}{2}}$. The discrete Fourier coefficients are

$$\xi^n(l) = \frac{1}{L} \int_0^L \rho^n(x) e^{-2\pi i l x / L} dx,$$

and the Parseval's equality for the discrete Fourier transform is

$$\|\rho^n\|_{l^2}^2 = \sum_{j=1}^{M-1} h |\rho_j^n|^2 = \int_0^L |\rho^n(x)|^2 dx = \sum_{l=-\infty}^{\infty} |\xi^n(l)|^2.$$

Suppose that the solution of the error equation has the following form $\rho_j^n = \xi^n e^{i\sigma j h}$, where $\sigma = 2\pi l / L$. Substituting ρ_j^n into error equations (10) and (11), we obtain

$$\xi^1 = \frac{\left(\frac{1}{\tau} + \mu\right) \left[(w_1 + w_3) \cos \sigma h + w_2 + i(w_1 - w_3) \sin \sigma h \right]}{(v_1 + v_3) \cos \sigma h + v_2 + i(v_1 - v_3) \sin \sigma h} \xi^0, \quad \text{for } n = 1, \quad (12)$$

and

$$\begin{aligned} \xi^n &= \frac{1}{(v_1 + v_3) \cos \sigma h + v_2 + i(v_1 - v_3) \sin \sigma h} \left[\frac{1}{\tau} \left[(w_1 + w_3) \cos \sigma h + w_2 + i(w_1 - w_3) \sin \sigma h \right] \xi^{n-1} \right. \\ &\quad + \mu \sum_{l=1}^{n-1} (a_{n-l-1} - a_{n-l}) \left[(w_1 + w_3) \cos \sigma h + w_2 + i(w_1 - w_3) \sin \sigma h \right] \xi^l \\ &\quad \left. + \mu a_{n-1} \left[(w_1 + w_3) \cos \sigma h + w_2 + i(w_1 - w_3) \sin \sigma h \right] \xi^0 \right], \quad \text{for } n \geq 2. \end{aligned} \quad (13)$$

Lemma 3.1. *The following inequality holds*

$$\left| \frac{\xi^n}{\xi^0} \right| \leq 1, \quad n = 1, 2, \dots, N. \quad (14)$$

Proof. We will prove the inequality using mathematical induction.

Step 1: When $n = 1$, we prove that $|\xi^1| \leq |\xi^0|$. For this, first we will verify that

$$\begin{aligned} & \frac{4\beta^2}{h^4}(1 - \cos \sigma h)^2 + 2\left(\frac{1}{\tau} + \mu\right) \left(\frac{2\beta_2}{h^2} \cos \sigma h + 1 - \frac{2\beta_2}{h^2}\right) \frac{2\beta}{h^2}(1 - \cos \sigma h) \\ & + 2\left(\frac{1}{\tau} + \mu\right) \frac{c\beta_1}{h^2} \sin^2 \sigma h + \frac{c^2}{h^2} \sin^2 \sigma h \geq 0. \end{aligned} \quad (15)$$

Consider the LHS of (15),

$$\begin{aligned} & \frac{4\beta^2}{h^4} \left(2 \sin^2 \frac{\sigma h}{2}\right)^2 + \left(\frac{1}{\tau} + \mu\right) \left(1 - \frac{4\beta_2}{h^2} \sin^2 \frac{\sigma h}{2}\right) \frac{8\beta}{h^2} \sin^2 \frac{\sigma h}{2} + 8\left(\frac{1}{\tau} + \mu\right) \frac{c\beta_1}{h^2} \sin^2 \frac{\sigma h}{2} \cos^2 \frac{\sigma h}{2} \\ & + \frac{4c^2}{h^2} \sin^2 \frac{\sigma h}{2} \cos^2 \frac{\sigma h}{2}. \end{aligned}$$

If $\sin^2 \frac{\sigma h}{2} = 0$, then (15) holds. Otherwise, divide the above expression by $\left(\frac{1}{\tau} + \mu\right) \frac{1}{h^2} \sin^2 \frac{\sigma h}{2}$, we get

$$\frac{16}{\left(\frac{1}{\tau} + \mu\right)} \frac{\beta^2}{h^2} \left(\sin^2 \frac{\sigma h}{2}\right) + \left(1 - \frac{4\beta_2}{h^2} \sin^2 \frac{\sigma h}{2}\right) 8\beta + 8c\beta_1 \cos^2 \frac{\sigma h}{2} + \frac{4}{\left(\frac{1}{\tau} + \mu\right)} c^2 \cos^2 \frac{\sigma h}{2}.$$

Since the first and last terms in the above summation are non-negative, it is sufficient to show that

$$8 \left(\beta \left(1 - \frac{4\beta_2}{h^2} \sin^2 \frac{\sigma h}{2}\right) + (d - \beta) \cos^2 \frac{\sigma h}{2} \right), \quad (16)$$

is non-negative. For convenience, we put $u = \frac{ch}{2d}$, then $\beta = du \coth(u)$ and

$$\beta_2 = \left(\frac{d}{c}\right)^2 (1 - u \coth(u)) + \frac{h^2}{6}. \text{ As } u \coth(u) \geq 1 \text{ holds, } \forall u \in (-\infty, +\infty), \text{ we get}$$

$$\begin{aligned} & \beta \left(1 - 4\beta_2 \frac{\sin^2 \left(\frac{\sigma h}{2}\right)}{h^2}\right) + (d - \beta) \cos^2 \left(\frac{\sigma h}{2}\right) = \beta \left(1 - 4\frac{\beta_2}{h^2}\right) \sin^2 \left(\frac{\sigma h}{2}\right) + d \cos^2 \left(\frac{\sigma h}{2}\right) \\ & \geq du \coth(u) \left(\frac{1}{3} + 4\left(\frac{d}{ch}\right)^2 (u \coth(u) - 1)\right) \sin^2 \left(\frac{\sigma h}{2}\right) \geq 0, \end{aligned}$$

thus (15) is true. Therefore, we have

$$\begin{aligned} & \left| \left(\frac{1}{\tau} + \mu\right) [(w_1 + w_3) \cos \sigma h + w_2 + i(w_1 - w_3) \sin \sigma h] \right| \leq |(v_1 + v_3) \cos \sigma h + v_2 + i(v_1 - v_3) \sin \sigma h|. \\ & \iff \left| \left(\frac{1}{\tau} + \mu\right) \left(2\frac{\beta_2}{h^2} \cos \sigma h + \left(1 - \frac{2\beta_2}{h^2}\right) + i\frac{\beta_1}{h} \sin \sigma h\right) \right| \\ & \leq \left| \left(2\left(\frac{1}{\tau} + \mu\right) \frac{\beta_2}{h^2} - \frac{2\beta}{h^2}\right) \cos \sigma h + \left(\frac{1}{\tau} + \mu\right) \left(1 - \frac{2\beta_2}{h^2}\right) + \frac{2\beta}{h^2} + i\left(\left(\frac{1}{\tau} + \mu\right) \frac{\beta_1}{h} + \frac{c}{h}\right) \sin \sigma h \right| \end{aligned}$$

$$\begin{aligned}
&\Longleftrightarrow \left(\frac{1}{\tau} + \mu\right)^2 \left[\left(2\frac{\beta_2}{h^2} \cos \sigma h + \left(1 - \frac{2\beta_2}{h^2}\right)\right)^2 + \left(\frac{\beta_1}{h} \sin \sigma h\right)^2 \right] \\
&\leq \left(\left(2\left(\frac{1}{\tau} + \mu\right) \frac{\beta_2}{h^2} - \frac{2\beta}{h^2}\right) \cos \sigma h + \left(\frac{1}{\tau} + \mu\right) \left(1 - \frac{2\beta_2}{h^2}\right) + \frac{2\beta}{h^2} \right)^2 + \left(\left(\left(\frac{1}{\tau} + \mu\right) \frac{\beta_1}{h} + \frac{c}{h}\right) \sin \sigma h \right)^2 \\
&\Longleftrightarrow \left(\frac{1}{\tau} + \mu\right)^2 \left(2\frac{\beta_2}{h^2} \cos \sigma h + 1 - \frac{2\beta_2}{h^2}\right)^2 + \left(\frac{1}{\tau} + \mu\right)^2 \left(\frac{\beta_1}{h} \sin \sigma h\right)^2 \\
&\leq \left[\left(\frac{1}{\tau} + \mu\right) \left(2\frac{\beta_2}{h^2} \cos \sigma h + 1 - \frac{2\beta_2}{h^2}\right) + \frac{2\beta}{h^2} (1 - \cos \sigma h) \right]^2 + \left(\left(\left(\frac{1}{\tau} + \mu\right) \frac{\beta_1}{h} + \frac{c}{h}\right) \sin \sigma h \right)^2.
\end{aligned}$$

This proves step 1.

Step 2: Assume that $|\xi^n| \leq |\xi^0|$ holds for $n = 1, 2, \dots, N-1$.

Step 3: We prove the inequality (14) for $n = N$. Using Lemma (2.2), we obtain

$$\begin{aligned}
|\xi^n| &\leq \frac{1}{|(v_1 + v_3) \cos \sigma h + v_2 + i(v_1 - v_3) \sin \sigma h|} \left[\frac{1}{\tau} \left| (w_1 + w_3) \cos \sigma h + w_2 + i(w_1 - w_3) \sin \sigma h \right| |\xi^{n-1}| \right. \\
&\quad + \mu \sum_{l=1}^{n-1} (a_{n-l-1} - a_{n-l}) \left| (w_1 + w_3) \cos \sigma h + w_2 + i(w_1 - w_3) \sin \sigma h \right| |\xi^l| \\
&\quad \left. + \mu a_{n-1} \left| (w_1 + w_3) \cos \sigma h + w_2 + i(w_1 - w_3) \sin \sigma h \right| |\xi^0| \right] \\
&\leq \frac{\frac{1}{\tau} \left| (w_1 + w_3) \cos \sigma h + w_2 + i(w_1 - w_3) \sin \sigma h \right|}{|(v_1 + v_3) \cos \sigma h + v_2 + i(v_1 - v_3) \sin \sigma h|} |\xi^0| \\
&\quad + \frac{\mu \left| (w_1 + w_3) \cos \sigma h + w_2 + i(w_1 - w_3) \sin \sigma h \right|}{|(v_1 + v_3) \cos \sigma h + v_2 + i(v_1 - v_3) \sin \sigma h|} |\xi^0| \\
&= \left(\frac{1}{\tau} + \mu\right) \frac{\left| (w_1 + w_3) \cos \sigma h + w_2 + i(w_1 - w_3) \sin \sigma h \right|}{|(v_1 + v_3) \cos \sigma h + v_2 + i(v_1 - v_3) \sin \sigma h|} |\xi^0| \leq |\xi^0|,
\end{aligned}$$

and therefore the mathematical induction is completed. $\square$

Theorem 3.2. The proposed exponential compact scheme (6)-(7) is unconditionally stable.

Proof. Using Lemma (3.1), we obtain

$$\begin{aligned}
\|\mathbf{Y}^n - \tilde{\mathbf{Y}}^n\|_{l^2}^2 &= \|\rho^n\|_{l^2}^2 = h \sum_{j=1}^{M-1} |\rho_j^n|^2 = \frac{h}{L} \sum_{j=1}^{M-1} |\xi^n e^{i\sigma j h}|^2 \\
&= \frac{h}{L} \sum_{j=1}^{M-1} |\xi^n|^2 \leq \frac{h}{L} \sum_{j=1}^{M-1} |\xi^0|^2 = \frac{h}{L} \sum_{j=1}^{M-1} |\xi^0 e^{i\sigma j h}|^2 \\
&= \|\rho^0\|_{l^2}^2 = \|\mathbf{Y}^0 - \tilde{\mathbf{Y}}^0\|_{l^2}^2, \quad n = 1, 2, \dots, N.
\end{aligned}$$

This completes the proof. $\square$

3.3 Convergence analysis

In this section, the convergence analysis of the proposed exponential compact numerical technique is studied using Fourier analysis approach. Denote the error in the numerical solution with $e_j^n = y_j^n - Y_j^n$, $1 \leq j \leq M-1$ and $1 \leq n \leq N$, then the error equation for $n = 1$ is

$$v_1 e_{j+1}^1 + v_2 e_j^1 + v_3 e_{j-1}^1 = \left(\frac{1}{\tau} + \mu \right) (w_1 e_{j+1}^0 + w_2 e_j^0 + w_3 e_{j-1}^0) + R_j^1, \quad (17)$$

and for $n \geq 2$, we have

$$v_1 e_{j+1}^n + v_2 e_j^n + v_3 e_{j-1}^n = \frac{1}{\tau} (w_1 e_{j+1}^{n-1} + w_2 e_j^{n-1} + w_3 e_{j-1}^{n-1}) + \mu \sum_{l=1}^{n-1} (a_{n-l-1} - a_{n-l}) (w_1 e_{j+1}^l + w_2 e_j^l + w_3 e_{j-1}^l) + \mu a_{n-1} (w_1 e_{j+1}^0 + w_2 e_j^0 + w_3 e_{j-1}^0) + R_j^n. \quad (18)$$

Using initial and boundary conditions we have $e_j^0 = 0$, $0 \leq j \leq M$ and $e_0^n = e_M^n = 0$, $1 \leq n \leq N$. The grid functions are defined as

$$e^n(x) = \begin{cases} e_j^n, & x_{j-\frac{h}{2}} < x \leq x_{j+\frac{h}{2}}, \quad 1 \leq j \leq M-1, \\ 0, & 0 \leq x \leq \frac{h}{2} \text{ and } L - \frac{h}{2} < x \leq L, \end{cases} \quad (19)$$

and

$$R^n(x) = \begin{cases} R_j^n, & x_{j-\frac{h}{2}} < x \leq x_{j+\frac{h}{2}}, \quad 1 \leq j \leq M-1, \\ 0, & 0 \leq x \leq \frac{h}{2} \text{ and } L - \frac{h}{2} < x \leq L. \end{cases} \quad (20)$$

Therefore, the Fourier series for $e^n(x)$ and $R^n(x)$ can be expressed as

$$e^n(x) = \sum_{l=-\infty}^{\infty} \eta^n(l) e^{2\pi i l x / L}, \quad R^n(x) = \sum_{l=-\infty}^{\infty} \zeta^n(l) e^{2\pi i l x / L}, \quad (21)$$

where

$$\eta^n(l) = \frac{1}{L} \int_0^L e^n(x) e^{-2\pi i l x / L} dx, \quad \zeta^n(l) = \frac{1}{L} \int_0^L R^n(x) e^{-2\pi i l x / L} dx.$$

Using the Parseval's equality and discrete l^2 norm, we have

$$\|e^n\|_{l^2}^2 = \sum_{n=1}^{M-1} h |e_j^n|^2 = \int_0^L |e^n(x)|^2 dx = \sum_{l=-\infty}^{\infty} |\eta^n(l)|^2, \quad (22)$$

$$\|R^n\|_{l^2}^2 = \sum_{n=1}^{M-1} h |R_j^n|^2 = \int_0^L |R^n(x)|^2 dx = \sum_{l=-\infty}^{\infty} |\zeta^n(l)|^2, \quad (23)$$

where

$$e^n = [e_1^n, e_2^n, \dots, e_{M-1}^n]^T \text{ and } R^n = [R_1^n, R_2^n, \dots, R_{M-1}^n]^T.$$

The series on the right hand side of the equation (23) is convergent, hence there exist a constant $K_2 > 0$, such that

$$|\zeta^n| = |\zeta^n(l)| \leq \frac{1}{3} K_2 |\zeta^1(l)| = \frac{1}{3} K_2 |\zeta^1|, \quad n = 1, 2, \dots, N. \quad (24)$$

Let $\sigma = 2\pi l / L$, we suppose that the errors have the following form $e_j^n = \eta^n e^{i\sigma j h}$, $R_j^n = \zeta^n e^{i\sigma j h}$. Substituting them into the error equations (17) and (18) and observing that $\eta_0 = 0$, we get

$$\eta^1 = \frac{\zeta^1}{(v_1 + v_3) \cos \sigma h + v_2 + i(v_1 - v_3) \sin \sigma h}, \quad \text{for } n = 1, \quad (25)$$

and

$$\eta^n = \frac{1}{(v_1 + v_3) \cos \sigma h + v_2 + i(v_1 - v_3) \sin \sigma h} \left[\frac{1}{\tau} \left[(w_1 + w_3) \cos \sigma h + w_2 + i(w_1 - w_3) \sin \sigma h \right] \eta^{n-1} \right. \\ \left. + \mu \sum_{l=1}^{n-1} (a_{n-l-1} - a_{n-l}) \left[(w_1 + w_3) \cos \sigma h + w_2 + i(w_1 - w_3) \sin \sigma h \right] \eta^l + \zeta^n \right], \quad \text{for } n \geq 2. \quad (26)$$

Lemma 3.2. *The following inequality holds for $0 < \tau < 1$,*

$$|(v_1 + v_3) \cos \sigma h + v_2 + i(v_1 - v_3) \sin \sigma h| \geq \frac{1}{3}. \quad (27)$$

Proof. For $\tau \in (0, 1)$, we obtain

$$\begin{aligned} & |(v_1 + v_3) \cos \sigma h + v_2 + i(v_1 - v_3) \sin \sigma h| \\ &= \left| \left(2 \left(\frac{1}{\tau} + \mu \right) \frac{\beta_2}{h^2} - \frac{2\beta}{h^2} \right) \cos \sigma h + \left(\frac{1}{\tau} + \mu \right) \left(1 - \frac{2\beta_2}{h^2} \right) + \frac{2\beta}{h^2} + i \left(\left(\frac{1}{\tau} + \mu \right) \frac{\beta_1}{h} + \frac{c}{h} \right) \sin \sigma h \right| \\ &= \left| \frac{2\beta}{h^2} (1 - \cos \sigma h) - \frac{2\beta_2}{h^2} \left(\frac{1}{\tau} + \mu \right) (1 - \cos \sigma h) + \left(\frac{1}{\tau} + \mu \right) + i \left(\left(\frac{1}{\tau} + \mu \right) \frac{\beta_1}{h} + \frac{p}{h} \right) \sin \sigma h \right| \\ &\geq \left| (1 - \cos \sigma h) \left(\frac{2\beta}{h^2} - \frac{2\beta_2}{h^2} \left(\frac{1}{\tau} + \mu \right) \right) + \left(\frac{1}{\tau} + \mu \right) \right| \\ &= \left| (1 - \cos \sigma h) \left(\frac{2\beta}{h^2} - 2 \frac{(\frac{1}{\tau} + \mu)}{h^2} \left(\frac{d^2}{c^2} (1 - u \coth u) + \frac{h^2}{6} \right) \right) + \left(\frac{1}{\tau} + \mu \right) \right| \\ &= \left| (1 - \cos \sigma h) \left(\frac{2\beta}{h^2} + 2 \frac{(\frac{1}{\tau} + \mu)}{h^2} \left(\frac{d^2}{c^2} (u \coth u - 1) \right) \right) + \left(\frac{1}{\tau} + \mu \right) \left(\frac{(\cos \sigma h - 1)}{3} + 1 \right) \right| \\ &\geq \left| \left(\frac{1}{\tau} + \mu \right) \left(\frac{(\cos \sigma h - 1)}{3} + 1 \right) \right| \geq \left| \frac{1}{\tau} \left(\frac{(\cos \sigma h - 1)}{3} + 1 \right) \right| \geq \frac{1}{\tau} \frac{1}{3} \geq \frac{1}{3}. \end{aligned}$$

□

Lemma 3.3. *The following inequality holds*

$$|\eta^n| \leq K_2(n\tau) |\zeta^1|, \quad n = 1, 2, \dots, N. \quad (28)$$

Proof. For $n = 1$, by means of Lemma (3.2) and the inequality (24), we have

$$|\eta^1| = \frac{|\zeta^1|}{|(v_1 + v_3) \cos \sigma h + v_2 + i(v_1 - v_3) \sin \sigma h|} \leq 3|\zeta^1| \leq 3 \frac{1}{3} K_2 \tau |\zeta^1| = K_2 \tau |\zeta^1|. \quad (29)$$

Suppose that

$$|\eta^j| \leq j K_2 \tau |\zeta^1| \quad \text{for } j = 1, 2, 3, \dots, n-1,$$

then, by using Lemmas (2.2), (3.2) and the inequality (24), we get

$$\begin{aligned}
 |\eta^n| &\leq \frac{1}{\left| (v_1 + v_3) \cos \sigma h + v_2 + i(v_1 - v_3) \sin \sigma h \right|} \left[\frac{1}{\tau} \left| (w_1 + w_3) \cos \sigma h + w_2 + i(w_1 - w_3) \sin \sigma h \right| |\eta^{n-1}| \right. \\
 &\quad \left. + \mu \sum_{l=1}^{n-1} (a_{n-l-1} - a_{n-l}) \left| (w_1 + w_3) \cos \sigma h + w_2 + i(w_1 - w_3) \sin \sigma h \right| |\eta^l| + |\zeta^n| \right] \\
 &\leq \frac{K_2((n-1)\tau) |\zeta^1|}{\left| (v_1 + v_3) \cos \sigma h + v_2 + i(v_1 - v_3) \sin \sigma h \right|} \left[\left| (w_1 + w_3) \cos \sigma h + w_2 \right. \right. \\
 &\quad \left. \left. + i(w_1 - w_3) \sin \sigma h \right| \left(\frac{1}{\tau} + \mu \sum_{l=1}^{n-1} (a_{n-l-1} - a_{n-l}) \right) \right] + \frac{|\zeta^n|}{\left| (v_1 + v_3) \cos \sigma h + v_2 + i(v_1 - v_3) \sin \sigma h \right|} \\
 &\leq \frac{K_2((n-1)\tau) |\zeta^1|}{\left| (v_1 + v_3) \cos \sigma h + v_2 + i(v_1 - v_3) \sin \sigma h \right|} \left[\left| (w_1 + w_3) \cos \sigma h + w_2 \right. \right. \\
 &\quad \left. \left. + i(w_1 - w_3) \sin \sigma h \right| \left(\frac{1}{\tau} + \mu \right) \right] + \frac{\frac{1}{3} K_2 \tau |\zeta^1|}{\left| (v_1 + v_3) \cos \sigma h + v_2 + i(v_1 - v_3) \sin \sigma h \right|} \\
 &\leq K_2((n-1)\tau) |\zeta^1| + K_2 \tau |\zeta^1| = K_2(n\tau) |\zeta^1|.
 \end{aligned}$$

The proof is completed. $\square$

Theorem 3.3. *The proposed compact exponential numerical scheme (6)-(7) is convergent with fourth order in space and first order in time.*

Proof. Denote $R_j^n = \mathcal{O}(\tau + h^4)$. For R_j^n , there exists a constant $K_3 > 0$ satisfying

$$|R_j^n| \leq K_3 (\tau + h^4). \quad (30)$$

Then, we obtain

$$\|R^n\|_{l^2}^2 = \sum_{j=1}^{M-1} h |R_j^n|^2 \leq L K_3^2 (\tau + h^4)^2. \quad (31)$$

Since $n\tau \leq T$ and using Lemma (3.3), we obtain

$$\begin{aligned}
 \|e^n\|_{l^2}^2 &= \sum_{l=-\infty}^{\infty} |\eta^n(l)|^2 \leq \sum_{l=-\infty}^{\infty} K_2^2(n\tau)^2 |\zeta^1|^2 = K_2^2(n\tau)^2 \sum_{l=-\infty}^{\infty} |\zeta^1|^2 \\
 &= K_2^2(n\tau)^2 \|R^1\|_{l^2}^2 \leq K_2^2(n\tau)^2 L K_3^2 (\tau + h^4)^2 \leq K_2^2 K_3^2 T^2 L (\tau + h^4)^2 \\
 &= K^2 (\tau + h^4)^2,
 \end{aligned}$$

where $K = K_2 K_3 T \sqrt{L}$, that is $\|e^n\|_{l^2} \leq K (\tau + h^4)$, which completes the proof. $\square$

4 Numerical Observations

In this section, we verify the accuracy and reliability of the derived scheme (6)-(7) by presenting some numerical examples for solving equation (1). The error in the numerical solution of the proposed scheme is computed by $Err(x, \tau) = \max_{1 \leq k \leq N} \|e^k\|_{l^2}$, where $\|e^k\|_{l^2} = \left(h \sum_{j=1}^{M-1} e_j^2 \right)^{\frac{1}{2}}$.

The computational order of convergence in the temporal and spatial direction is computed using $\log_2 \left(\frac{Err(h, 2\tau)}{Err(h, \tau)} \right)$ and $\log_2 \left(\frac{Err(2h, \tau)}{Err(h, \tau)} \right)$ respectively.

4.1 Example 1

Consider the problem,

$$\begin{cases} \frac{\partial y}{\partial t} + 0.5 \frac{\partial^\gamma y}{\partial t^\gamma} + \frac{\partial y}{\partial x} - 2 \frac{\partial^2 y}{\partial x^2} = t^\gamma (1 + \gamma) \sin 2x + (8 + 8t^{1+\gamma}) \sin 2x + (2 + 2t^{1+\gamma}) \cos 2x \\ \qquad \qquad \qquad + \frac{\Gamma(2 + \gamma)}{2} t \sin 2x, \\ y(0, t) = 0, \qquad y(1, t) = \sin 2 + t^{1+\gamma} \sin 2, \quad t > 0, \\ y(x, 0) = \sin 2x, \quad 0 \leq x \leq 1, \end{cases}$$

where the analytical solution is $y(x, t) = (1 + t^{1+\gamma}) \sin 2x$. At first, we consider a spatial step $h = 1/1000$ and $\gamma = 0.3, 0.5, 0.7$ to check the numerical order of accuracy with respect to time variable. Table 1 reveals the first order accuracy in time and the corresponding errors. Secondly, the numerical order of accuracy of the scheme in x -direction is evaluated with a temporal step size $\tau = 1/200000$. Table 2 gives the errors and spatial order of convergence. The results shows that the proposed method achieved fourth order accuracy in space that matches with the theoretical order. Figure 1 reveals good agreement between the numerical and analytical solutions. Figure 2 depicts the space-time graph of Example 1 at $T = 1$ with $\gamma = 0.1$ and $\gamma = 0.9$.

Table 1: Errors and temporal convergence order at $T = 1$ for Example 1.

τ	$\gamma = 0.3$		$\gamma = 0.5$		$\gamma = 0.7$	
	error	order	error	order	error	order
1/100	6.0658e-04		5.0522e-04		4.1410e-04	
1/200	3.3748e-04	0.8459	2.6493e-04	0.9313	2.0817e-04	0.9922
1/400	1.8350e-04	0.8790	1.3693e-04	0.9522	1.0406e-04	1.0004

Table 2: Errors and spatial convergence order at $T = 0.1$ for Example 1.

h	$\gamma = 0.3$		$\gamma = 0.5$		$\gamma = 0.7$	
	error	order	error	order	error	order
1/3	2.6894e-04		2.5976e-04		2.4977e-04	
1/6	1.6961e-05	3.9870	1.6367e-05	3.9883	1.5707e-05	3.9911
1/12	1.3125e-06	3.6918	1.2509e-06	3.7098	1.1674e-06	3.7500

4.2 Example 2

Consider the problem,

$$\begin{cases} \frac{\partial y}{\partial t} + \frac{\partial^\gamma y}{\partial t^\gamma} + \frac{\partial y}{\partial x} - \frac{\partial^2 y}{\partial x^2} = \left(1 + \frac{t^{1-\gamma}}{\Gamma(2-\gamma)} \right) \sin(\pi x) + \left(\pi t \cos(\pi x) + \pi^2 t \sin(\pi x) \right), \\ y(0, t) = 0, \quad y(1, t) = 0, \quad t > 0, \\ y(x, 0) = 0, \quad 0 \leq x \leq 1, \end{cases}$$

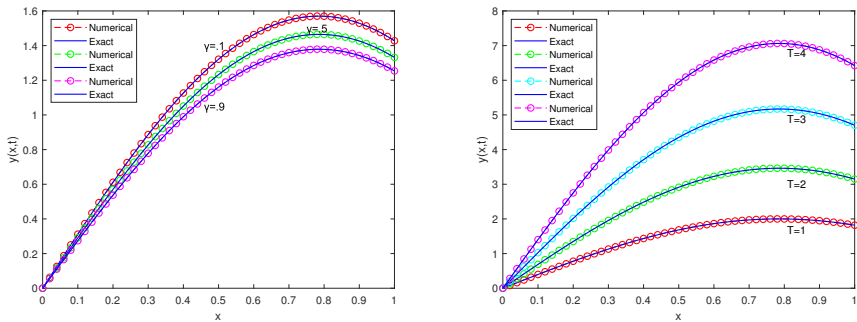


Figure 1: Numerical solution of Example 1 for various values of γ with $h = 1/50$ and $\tau = 1/100$ at time $T = 0.6$ (left). Numerical solution of Example 1 for various values of time T with $h = 1/50$, $\tau = 1/100$ and $\gamma = 0.3$ (right).

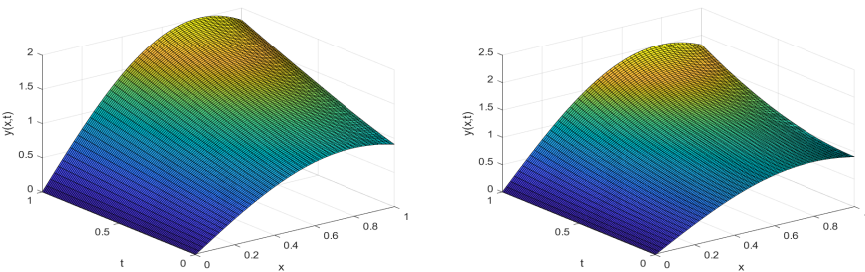


Figure 2: Space-time graphs of Example 1 with $h = 1/50$, $\tau = 1/100$ and $T = 1$ for $\gamma = 0.1$ (left) and $\gamma = 0.9$ (right).

where the analytical solution is $y(x, t) = t \sin(\pi x)$. Table 3 displays the errors and computational order of convergence for various values of h with $\tau = 1/50$. It is evident from the Table 3 that the desired computational order is achieved. Figure 3 presents the plots of the numerical and analytical solutions of Example 2 at different time levels and the space-time graph framed at $T = 1$ for $\gamma = 0.5$. It is evident that the numerical results are completely consistent with the analytical results.

Table 3: Errors and convergence order for Example 2 with $\tau = 1/50$, $T = 1$.

h	$\gamma = 0.3$		$\gamma = 0.5$		$\gamma = 0.7$	
	error	order	error	order	error	order
1/4	7.9112e-04		7.8541e-04		7.8122e-04	
1/8	4.9082e-05	4.0106	4.8728e-05	4.0106	4.8470e-05	4.0106
1/16	3.0612e-06	4.0030	3.0392e-06	4.0030	3.0231e-06	4.0030
1/32	1.9122e-07	4.0008	1.8985e-07	4.0008	1.8884e-07	4.0008
1/ 64	1.1950e-08	4.0002	1.1864e-08	4.0002	1.1801e-08	4.0002
1/128	7.4670e-10	4.0003	7.4152e-10	4.0000	7.3726e-10	4.0006
1/256	4.7146e-11	3.9853	4.5593e-11	4.0236	4.5633e-11	4.0140

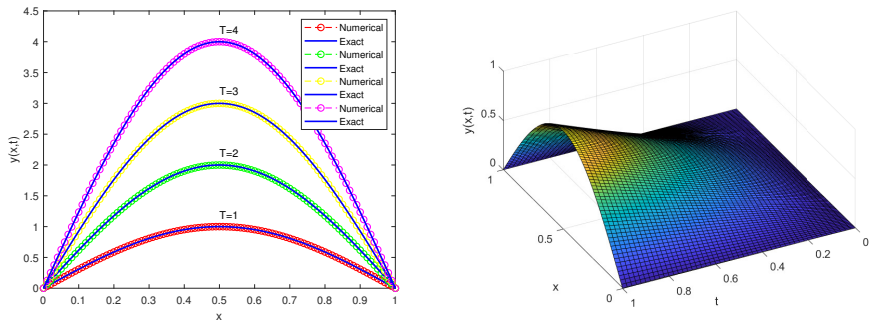


Figure 3: Numerical and exact solutions of Example 2 for various values of time T with $h = 1/100$, $\tau = 1/50$ and $\gamma = 0.3$ (left). Space-time graph of Example 2 with $h = 1/100$, $\tau = 1/50$, $\gamma = 0.5$ and $T = 1$ (right).

4.3 Example 3

Consider the problem,

$$\begin{cases} \frac{\partial y}{\partial t} + \frac{\partial^\gamma y}{\partial t^\gamma} + \frac{\partial y}{\partial x} - \frac{\partial^2 y}{\partial x^2} = \left[\frac{t^{1-\gamma}}{\Gamma(2-\gamma)} - \frac{2(tx-t\lambda)}{\alpha} + \frac{2t}{\alpha} - \frac{4t(x-\lambda)^2}{\alpha^2} + 1 \right] \exp\left(-\frac{(x-\lambda)^2}{\alpha}\right), \\ y(0,t) = t \exp\left(\frac{-\lambda^2}{\alpha}\right), \quad y(1,t) = t \exp\left(-\frac{(1-\lambda)^2}{\alpha}\right), \quad t > 0, \\ y(x,0) = 0, \quad 0 \leq x \leq 1, \end{cases}$$

where the analytical solution is $y(x,t) = t \exp\left(-\frac{(x-\lambda)^2}{\alpha}\right)$. The numerical errors for Example 3 with $\tau = 1/100$, $\lambda = 0.04$, $\alpha = 0.1$ for different values of h and γ are presented in Table 4. Similarly, the numerical errors for Example 3 with $\tau = 1/100$, $\lambda = 0.3$, $\alpha = 1$ for different values of h and γ are presented in Table 5. The results given in Tables 4 and 5 indicates that the scheme is fourth order convergent and highly accurate. Figure (4) illustrates the numerical and analytical solution of Example 3 at different times with $\alpha = 0.001$, 0.1 and 1 , whereas, Figure (5) depicts the numerical and analytical solutions of Example 3 at different times with different $\lambda = 0.01, 0.1$ and 1 . These plots ensures a good correlation between analytical and numerical solutions. The space-time graphs are shown in Figure (6) and Figure (7) for different values of α and λ .

Table 4: Errors and convergence order for Example 3 with $\lambda = 0.04$, $\alpha = 0.1$ and $T = 2$.

h	τ	$\gamma = 0.1$		$\gamma = 0.5$		$\gamma = 0.7$	
		error	order	error	order	error	order
1/4	1/100	3.8199e-02		3.8521e-02		3.8753e-02	
1/8	1/100	1.7823e-03	4.4217	1.7978e-03	4.4213	1.8091e-03	4.4210
1/16	1/100	1.0647e-04	4.0652	1.0740e-04	4.0652	1.0807e-04	4.0652
1/32	1/100	6.5816e-06	4.0159	6.6391e-06	4.0159	6.6807e-06	4.0158
1/64	1/100	4.1023e-07	4.0039	4.1381e-07	4.0039	4.1641e-07	4.0039

Table 5: Errors and convergence order for Example 3 with $\lambda = 0.3, \alpha = 1$ and $T = 1$.

h	τ	$\gamma = 0.1$		$\gamma = 0.5$		$\gamma = 0.7$	
		error	order	error	order	error	order
1/4	1/100	3.6551e-05		3.6411e-05		3.6393e-05	
1/8	1/100	2.2952e-06	3.9932	2.2864e-06	3.9932	2.2852e-06	3.9933
1/16	1/100	1.4300e-07	4.0045	1.4245e-07	4.0045	1.4237e-07	4.0046
1/32	1/100	8.9285e-09	4.0015	8.8937e-09	4.0015	8.8888e-09	4.0015
1/64	1/100	5.5788e-10	4.0004	5.5571e-10	4.0004	5.5541e-10	4.0004

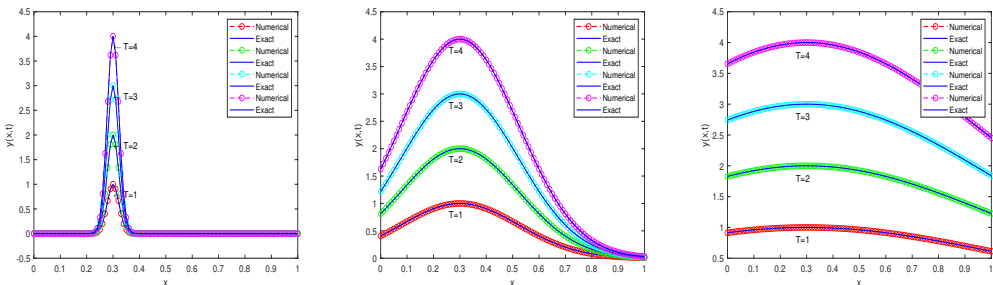


Figure 4: Numerical and exact solutions of Example 3 for various values of time T with $h = 1/100, \tau = 1/100, \gamma = 0.3$ and $\lambda = 0.3$ for different $\alpha = 0.001, 0.1, 1$ respectively.

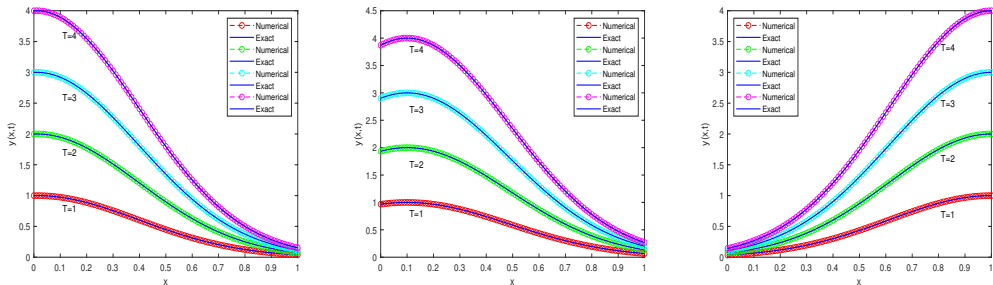


Figure 5: Numerical and exact solutions of Example 3 for various values of time T with $h = 1/100, \tau = 1/100, \gamma = 0.3$ and $\alpha = 0.3$ for $\lambda = 0.01, 0.1, 1$ respectively.

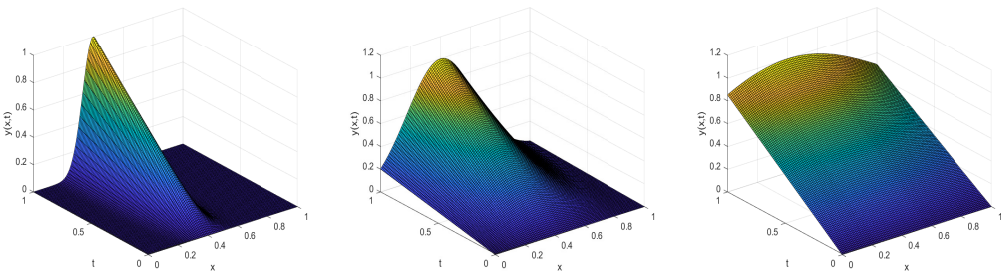


Figure 6: Space-time graphs of Example 3 with $h = 1/100, \tau = 1/100, \gamma = 0.7, T = 1$ and $\lambda = 0.4$ for $\alpha = 0.01, 0.1, 1$ respectively.

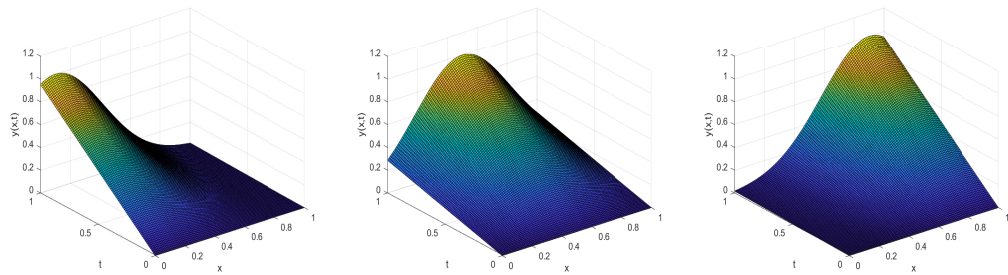


Figure 7: Space-time graphs of Example 3 with $h = 1/100$, $\tau = 1/100$, $\gamma = 0.7$, $T = 1$ and $\alpha = 0.2$ for $\lambda = 0.1, 0.5, 0.9$ respectively.

4.4 Example 4

Consider the problem,

$$\begin{cases} \frac{\partial y}{\partial t} + \frac{\partial^{\gamma(x,t)} y}{\partial t^{\gamma(x,t)}} + \frac{\partial y}{\partial x} - \frac{\partial^2 y}{\partial x^2} = 10(x - x^2)^2 + \frac{10(x - x^2)^2 t^{1-\gamma(x,t)}}{\Gamma(2 - \gamma(x,t))} + 20(t + 1)(x - 3x^2 + 2x^3) \\ \qquad \qquad \qquad - 20(t + 1)(1 - 6x + 6x^2), \\ y(0, t) = 0, \qquad \qquad \qquad y(1, t) = 0, \qquad t > 0, \\ y(x, 0) = 10(x - x^2)^2, \qquad 0 \leq x \leq 1. \end{cases}$$

Let $\gamma(x, t) = 1 - \frac{e^{-xt}}{2}$, then the analytical solution is $y(x, t) = 10(x - x^2)^2(t + 1)$. The absolute errors are provided in Table 6 with $h = \tau = 1/100$. Table 7 displays the convergence rates at $T = 1$. The left of Figure 8 depicts the solution behaviour for several T values and the right shows the space-time graph at $T = 1$.

Table 6: Absolute error at $T = 1$ of Example 4.

x	Numerical solution (Proposed Method)	Exact solution	Absolute error [33]	Absolute error (Proposed Method)
0.1	0.161999991425576	0.162000000000000	0.00015629	0.000000008574424
0.2	0.511999982503170	0.512000000000000	0.0014006	0.000000017496830
0.3	0.881999979160857	0.882000000000000	0.00297519	0.000000020839142
0.4	1.151999985510861	1.152000000000000	0.00429766	0.000000014489139
0.5	1.250000000653446	1.250000000000000	0.00497219	0.000000006534461
0.6	1.152000019190956	1.152000000000000	0.00480341	0.000000019190955
0.7	0.882000033474847	0.882000000000000	0.00381527	0.000000033474846
0.8	0.512000036441107	0.512000000000000	0.00227469	0.000000036441107
0.9	0.162000024461538	0.162000000000000	0.00072075	0.000000024461538

Table 7: Errors and convergence order for Example 4 with $\tau = 1/100$ at $T = 1$.

h	error	order
1/4	1.2291e-03	
1/8	7.6112e-05	4.0133
1/16	4.6908e-06	4.0202

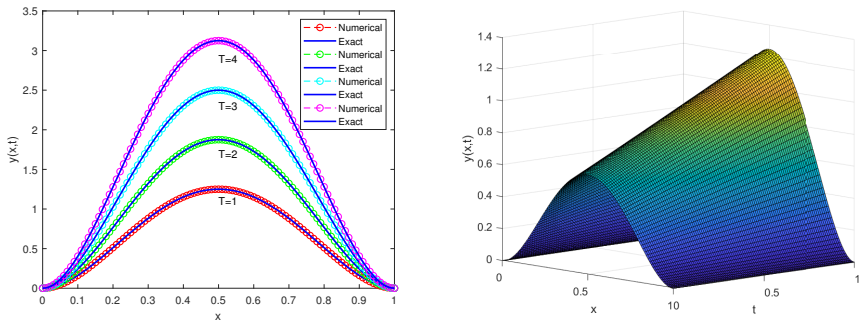


Figure 8: Numerical and exact solutions of Example 4 for various values of time T with $h = 1/100, \tau = 1/100$ (left). Space-time graph of Example 4 with $h = 1/100, \tau = 1/100$ and $T = 1$ (right).

4.5 Example 5

Consider the problem,

$$\begin{cases} \frac{\partial y}{\partial t} + \frac{\partial \gamma(x,t)}{\partial t} y + \frac{\partial y}{\partial x} - \frac{\partial^2 y}{\partial x^2} = \frac{5(x - x^2)t^{1-\gamma(x,t)}}{\Gamma(2 - \gamma(x,t))} + (5x - 5x^2) + (5t + 5)(1 - 2x) + 10t + 10, \\ y(0, t) = 0, \quad y(1, t) = 0, \quad t > 0, \\ y(x, 0) = (5x - 5x^2), \quad 0 \leq x \leq 1. \end{cases}$$

Let $\gamma(x, t) = \frac{4}{5} + \frac{1}{200} \sin(x) \cos(xt)$, then the analytical solution is $y(x, t) = 5(x - x^2)(t + 1)$. Table 8 provides the absolute error with $h = \tau = 1/100$. In Figure 9, the analytical and numerical solutions are compared, and also space-time graph is provided.

Table 8: Absolute error at $T = 1$ of Example 5.

x	Numerical solution	Exact solution	Absolute error
0.1	0.899999999063245	0.900000000000000	0.000000000936754
0.2	1.599999998963959	1.600000000000000	0.000000010360408
0.3	2.099999999372451	2.100000000000000	0.000000000627548
0.4	2.39999999984538	2.400000000000000	0.00000000015461
0.5	2.500000000548305	2.500000000000000	0.000000000548304
0.6	2.400000000886606	2.400000000000000	0.000000000886605
0.7	2.100000000915478	2.100000000000001	0.000000000915477
0.8	1.600000000658977	1.600000000000000	0.000000000658977
0.9	0.900000000261597	0.900000000000000	0.000000000261596

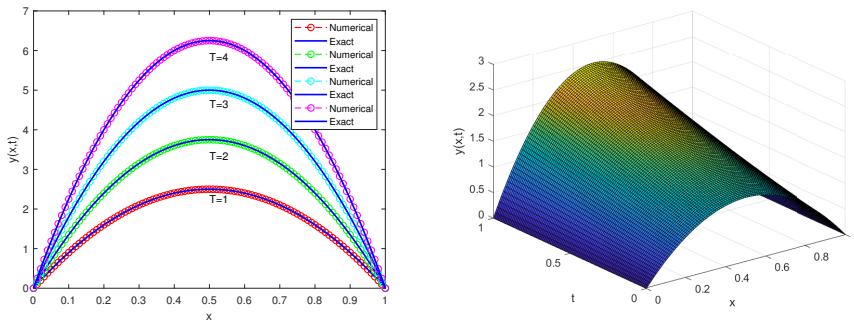


Figure 9: Exact and numerical solutions of Example 5 for various values of time T with $h = 1/100$, $\tau = 1/100$ (left). Space-time graph of Problem 5 with $h = 1/100$, $\tau = 1/100$ and $T = 1$ (right).

All the computations are performed in MATLAB R2020b on an AMD Ryzen 7 3700U 2.30 GHz CPU machine with 8 GB RAM. The outcomes shown in the tables demonstrate that the numerical and exact solutions are well-aligned. Table 6 clearly shows that the current method is highly accurate for solving mobile-immobile TFCDE with variable time fractional order. Also, the precision of the computations produced by the presented scheme is good and compatible with the theoretical order. The plots detailed for each example shows how coherently the newly proposed scheme works for solving equation (1).

5 Conclusions

In this work, we constructed a new higher order compact difference scheme for finding the numerical solution of variable order time fractional mobile-immobile convection-diffusion equation and performed its mathematical analysis rigorously. The new method is developed with the effective combination of backward Euler for the integer time derivative, a fourth order exponential compact difference approximation for the spatial derivatives and a Caputo time fractional derivative of order $(2 - \gamma)$ for the fractional time derivative. We have shown that the derived scheme is unconditionally stable, convergent and uniquely solvable. Through our analysis, we were able to demonstrate that the suggested scheme has first order temporal and fourth order spatial accuracy. Various numerical experiments were conducted to confirm the theoretical analysis and test the accuracy of the new scheme.

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Conflicts of Interest The authors declare no conflict of interest.

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